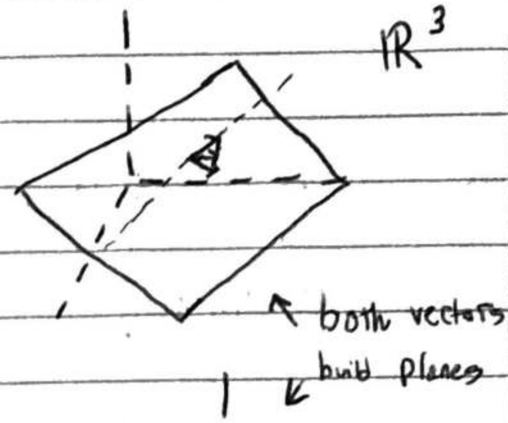
9/1/21 Notes

Last time: Dot Product

$\vec{u}$ and $\vec{v}$ are orthagonal if $\vec{u} \cdot \vec{v} = 0$

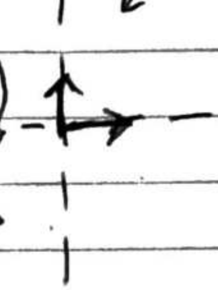
12.4 Cross Product



goal: give two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$. Construct a vector $\vec{w} = \langle w_1, w_2, w_3 \rangle \in \mathbb{R}^3$ so that $\vec{w}$ is orthoganal to $\vec{u}$ and $\vec{v}$

(want to find $\vec{w}$ canonically)

How?: we know that
$$\begin{cases} ① \; 0 = \vec{u} \cdot \vec{w} = (u_1 w_1 + u_2 w_2 + u_3 w_3) \\ ② \; 0 = \vec{v} \cdot \vec{w} = (v_1 w_1 + v_2 w_2 + v_3 w_3) \end{cases}$$



Give "this formula" we want to find $\langle w_1, w_2, w_3 \rangle = \vec{w}$

Therefore, we multiply ① by v_3 and ② by u_3 to obtain:
$$\begin{cases} 1^* \; 0 = v_3 (\vec{u} \cdot \vec{w}) = (u_1 v_3) w_1 + (u_2 v_3) w_2 + (u_3 v_3) w_3 \\ 2^* \; 0 = u_3 (\vec{v} \cdot \vec{w}) = (u_3 v_1) w_1 + (u_3 v_2) w_2 + (u_3 v_3) w_3 \end{cases}$$

Next subtract 2^* from 1^*

$$(A) \quad 0 = v_3(\vec{u} \cdot \vec{w}) - u_3(\vec{v} \cdot \vec{w})$$
$$= (u_1 v_3 - u_3 v_1) w_1 + (u_2 v_3 - u_3 v_2) w_2$$
$$= -(-(u_1 v_3 - u_3 v_1)) w_1 + (u_2 v_3 - u_3 v_2) w_2$$

aside: $-ax + by = 0$ has solution $\begin{pmatrix} x = b \\ y = a \end{pmatrix}$ to $-ab + ba = 0$

Hence: (A) has at least the solution
$$\begin{cases} w_1 = u_2 v_3 - u_3 v_2 \\ w_2 = -(u_1 v_3 - u_3 v_1) \end{cases}$$

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Inputting these to ① we obtain

$$0 = u_1 w_1 + u_2 w_2 + u_3 w_3$$
$$= u_1(u_2 v_3 - u_3 v_2) + u_2(-(u_1 v_3 - u_3 v_1)) + u_3 w_3$$
$$= u_1 u_2 v_3 - u_1 u_3 v_2 - u_1 u_2 v_3 + u_2 v_3 v_1 + u_3 w_3$$
$$= u_3(u_2 v_1 - u_1 v_2 + w_3)$$

Side note: either $u_3 = 0$ or $w_3 = u_1 v_2 - u_2 v_1$

Claim: (modulo the detail that u_3 may be 0)

<u>We have the solution:</u>

$\vec{w} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$

<u>Now Check ↑ symbolically</u>

<u>Def:</u> The determinant of the 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = +ad - bc$

Def: The determinant of the 3x3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

alteration in signs from 2x2 to 3x3 ... +, -, +, -, +, -, +

* when finding the formula dont worry about anything in the #'s row or column. 3x3 matrix is left with a $\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$. (Inside matrix are #'s or letters that are not in a's row/column.

ex: Compute determinate $\begin{bmatrix} -1 & 3 & 7 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

side note: always starts +

① $$\det = (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 7\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

② $$\det = (-1)\Big((-1(1)) - 1(0)\Big) - 3\Big((0(1)) - (1 \cdot 1)\Big) + 7\Big((0 \cdot 0) - (-1)(1)\Big)$$

$$= -1(-1) - 3(-1) + 7(1) = \boxed{11}$$

Why? (Find determinate of 3x3, then find determinate of 2x2, solve)

Def: let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

The cross product of $\vec{u}$ with $\vec{v}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\det = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \vec{i}\Big(u_2 v_3 - u_3 v_2\Big) - \vec{j}\,\Big|u_1 v_3 - u_3 v_1\Big| + \vec{k}\,\Big|u_1 v_2 - u_2 v_1\Big|$$

$$= \langle u_2 v_3 - u_3 v_2 \quad -u_1 v_3 - v_3 v_1 \quad u_1 v_2 - u\, v_1 \rangle$$

Resultant of Cross Product

NB: This has been done in $\mathbb{R}^3$. This only works in $\mathbb{R}^3$ (cross product)

The cross product as a vector operation
(vector in $\mathbb{R}^3$ x vector in $\mathbb{R}^3 \longrightarrow$ Vector in $\mathbb{R}^3$)

☆ $\vec{0} \times 1$ = Undefined $\rightarrow$ 1 is not in $\mathbb{R}^3$

☆ $\langle 1,1 \rangle \times \langle 3,2 \rangle$ = Undefined $\rightarrow$ not defined in $\mathbb{R}^3$

Prop (Algebraic Properties of Cross product):
let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

① $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Proof: $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

$$= \vec{i}\begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} - \vec{j}\begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} + \vec{k}\begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}$$

$$= \vec{i}(v_2u_3 - v_3u_2) - \vec{j}(v_1u_3 - v_3u_1) + \vec{k}(v_1u_2 - v_2u_1)$$

$$= \langle v_2u_3 - v_3u_2, -v_1u_3 - v_3u_1, v_1u_2 - v_2u_1 \rangle$$

$$= \langle -(u_2v_3 - u_3v_2), -(-(u_1v_3 - u_3v_1)), -(u_1v_2 - u_2v_1) \rangle$$

$$= -\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle$$

$$= -\vec{u} \times \vec{v}$$

Scalar vector

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$ communative

③ $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ distributive on left

④ $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$ distributive on right

☆ ⑤ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ cross product of cross product

prop (geometric properties of cross product)

let $\vec{u}, \vec{v} \in \mathbb{R}^3$

① $\vec{u} \times \vec{v}$ is orthoganal to both $\vec{u}$ and $\vec{v}$

② $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$ θ is the angle between $\vec{u}$ and $\vec{v}$

③ $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ is parralel to $\vec{v}$